Optimization of local control of chaos by an evolutionary algorithm

Hendrik Richter, Kurt J. Reinschke

Abstract

An evolutionary algorithm for optimizing local control of chaos is presented. Based on a Lyapunov approach, a linear control law and the state-space region in which this control law is activated are determined. In addition, we study a relation between certain adjustable design parameters and a particular measure of the uncontrolled chaotic attractor in the state-space region of control (SSRC). From this relation the objective function to be optimized is derived. In that context, we assume a linear control law to be given and optimize size and shape of the SSRC using an evolutionary algorithm. It is shown by examples how the algorithm can also be applied to higher-dimensional systems with possibly more than one positive Lyapunov exponent.

Keywords: Control of chaos; Optimization; Evolutionary algorithm

1. Introduction

Controlling chaos has been an important issue in nonlinear dynamics research for a number of years. In particular, approaches to local control became a matter of considerable interest (see, e.g., [1–6]). The idea of local control is to employ feedback only locally within a bounded region of state space, which we will call the state-space region of control (SSRC). In the more recent past, attempts have been made to calculate the SSRC to obtain a proof of stability for the controlled nonlinear system. Paskota [4] used ideas from functional analysis and Richter and Reinschke [6] proposed a general Lyapunov approach. This Lyapunov approach provides a proof of stability and a procedure for finding the SSRC. So, both the feedback law and the SSRC can be determined by this controller design method.

A drawback of all design procedures known to us for local control is that the calculated SSRC may be small compared with the real basin of attraction about the stabilized equilibrium point. As a consequence, the time...
elapsed before the control is achieved may be considerable. Thus, an important goal in optimizing local control
is an enlargement of the SSRC [7,8]. Unfortunately, a large SSRC does not necessarily imply a fast stabilization,
in particular if the overlap between chaotic attractor and SSRC is small. We therefore propose a measure which
characterizes the relative sojourn of a trajectory on the chaotic attractor within the SSRC. This measure is taken
as the objective function to be maximized. In other words, we maximize the amount of time a chaotic attractor
trajectory spends within the SSRC, rather than maximizing the intersection between chaotic attractor and SSRC. In
our formulation, the objective function depends on adjustable parameters which are optimized using an evolutionary
algorithm similar to the ones proposed in [9,10]. Recently, calculation of optimal feedback law for local control has
been studied [2,5]. We assume a feedback law (which might be the result of such a preliminary optimization of the
feedback law) to be given and optimize size and shape of the SSRC. So, the proposed method can also be seen as a
quasi-optimal calculation of the region of stability for a linear controller with given feedback gain matrix.

The paper is organized as follows. In Section 2 the Lyapunov approach to local control of chaos is summarized.
There we shall restrict ourselves to chaotic discrete-time systems and the problem of stabilizing equilibrium points.
As was shown in [6], continuous-time systems and stabilization of higher periodic orbits can be treated using the
same theoretical framework. Furthermore, the question of how to apply the approach in the absence of a priori
mathematical system model is addressed. Next, in Section 3, we introduce the objective function to be maximized
and present the evolutionary algorithm used to solve the optimization problem. The relation of the proposed method
to targeting is briefly discussed. In Section 4 two examples are considered: the Hénon map and a four-dimensional
generalization of the Hénon map with three positive Lyapunov exponents. Finally, in Section 5, a summary of the
findings is given.

The following notation is used throughout. \( \mathbb{R}^n \) denotes the \( n \)-dimensional Euclidean space with norm \( \| \cdot \| \); \( \mathbb{R}^{n \times m} \)
the set of all real \( n \times m \)-matrices whose induced norm is also denoted by \( \| \cdot \| \); \( \text{eig}_i(A) \), \( i = 1, 2, \ldots, n \) is the \( i \)th
eigenvalue of \( A \in \mathbb{R}^{n \times n} \); \( \det A \) is the determinant of a matrix \( A \); \( A^T \) is the transpose of \( A \) and if \( A \) is positive-definite,
we write \( A > 0 \).

2. Lyapunov approach to local control

2.1. Description of the design procedure

Consider a system described by the difference equation

\[
x(k + 1) = f(x(k), u(k)),
\]

where \( x(k) \in \mathbb{R}^n \) is the state vector, \( u(k) \in \mathbb{R}^m \) is the input vector and \( k = 0, 1, 2, \ldots \) We assume that the
mapping \( f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \) is analytic and that the system (1) displays chaotic behavior for a constant input
\( u(k) = \bar{u} \). Furthermore, let the initial state \( x(0) := x_0 \) of (1) belong to the basin of attraction \( B_A(A_R) \subseteq \mathbb{R}^n \) of the
chaotic attractor \( A_R \), and let \( \bar{x} \) denote an unstable equilibrium point such that \( f(\bar{x}, \bar{u}) - \bar{x} = 0 \). Our objective is to
stabilize the unstable equilibrium point \( \bar{x} \) by means of local feedback control. To do so, we define a feedback matrix
\( G \in \mathbb{R}^{m \times n} \) and a SSRC \( W \subset \mathbb{R}^n \), which contains the equilibrium point \( \bar{x} \).

We call a control scheme local control if the system (1) is driven by

\[
u(k) = \bar{u} - G(x(k) - \bar{x}) \quad \text{for} \quad x(k) \in W,
\]

\[
u(k) = \bar{u} = \text{constant} \quad \text{for} \quad x(k) \notin W.
\]
Thus, the controller is specified by the feedback gain matrix $G$ and the SSRC $W$. Both $G$ and $W$ may be considered as parameters to be determined by the controller design procedure outlined below. Fig. 1 shows the structure of the local control.

If Eq. (1) describing the dynamics of the system is not explicitly known, then we assume that a sufficiently long time series of the system’s state $x(k)$ with given input $u(k)$ is available. In that case, a system equation can be constructed using the delay coordinate embedding technique (see, e.g., [2,11,12] for an application to controlling chaos). In general, the obtained system equation has the form

$$ x(k + 1) = g(x(k), u(k), u(k - 1), u(k - 2), \ldots, u(k - l)), $$

i.e., the state $x(k + 1)$ does not only depend on $u(k)$ but also on the previous inputs $u(k - 1), u(k - 2), \ldots, u(k - l)$ for some $l \in \mathbb{N}$. Eq. (4) can be converted into an equation of the form (1) by defining the new state vector

$$ v(k + 1) = \begin{pmatrix} x(k + 1) \\ u(k) \\ u(k - 1) \\ \vdots \\ u(k - l + 1) \end{pmatrix}. $$

Hence, the methodology described here is, at least in principle, applicable in experimental situations where only a measured time series is available.

In the Lyapunov approach to local control the SSRC $W$ is determined, using a Lyapunov function, such that $W$ belongs to the basin of attraction of the stabilized equilibrium point $\bar{x}$. Then the equilibrium point $\bar{x}$ appears as a stable fixed point for all initial states $x_0 \in W$. So, we can state the following stability result for local control. Let $x_T(k, x_0)$ denote the transient from an initial state $x_0 \in B_A(A_R)$ to the chaotic attractor $A_R$. Then the condition

$$ W \cap (A_R \cup x_T) \neq \emptyset $$

is necessary and sufficient for stabilizing the previously unstable equilibrium point $\bar{x}$ by local control.

The state feedback control law is derived as follows. First expand $f(x, u)$ into a Taylor series about $(\bar{x}, \bar{u})$, insert the feedback (2), and define the new state variable $w(k) = x(k) - \bar{x}$ to obtain the closed-loop system

$$ w(k + 1) = (A - BG)w(k) + h(w(k)) \quad \text{for} \quad w(k) \in W, $$

where $A = \partial f / \partial x |_{x=\bar{x}, u=\bar{u}}$, $B = \partial f / \partial u |_{u=\bar{u}, x=\bar{x}}$, and $W$ denotes the SSRC in the $w$-space. The map $h(w) : \mathbb{R}^n \to \mathbb{R}^n$ comprises the terms of second and higher order of the Taylor series expansion.
Eq. (7) can be formally rewritten as

\[ w(k+1) = H_L w(k) + H_N(w(k)) \cdot w(k) \quad \text{for} \quad w(k) \in W, \tag{8} \]

where \( H_L = A - BG \), and \( H_N(w(k)) \) is defined such that \( H_N(w(k)) \cdot w(k) = h(w(k)) \) and \( H_N(0) = 0 \).

The system (1) is assumed to be locally controllable at \( \tilde{x} \); i.e., the matrix pair \((A, B)\) is controllable. Then there exists a feedback matrix \( G \) such that the eigenvalues of \( H_L = A - BG \) are placed arbitrarily (see, e.g., [13]). To ensure stability, the matrix \( G \) must be chosen such that all the eigenvalues of \( H_L \) have modulus less than one. The resulting closed-loop system described by (7) has a locally stable equilibrium point at \( w = 0 \). Hence, there exists a Lyapunov function \( V(w(k)) \) defined in some neighborhood \( W \) of \( w = 0 \) (see, e.g., [14], p. 301). Assume this Lyapunov function to be the quadratic form

\[ V(w(k)) = w(k)^T P w(k), \tag{9} \]

where \( P = P^T > 0 \). Note that such a Lyapunov function results in an elliptic shape of the SSRC. Provided

\[ \Delta V(w(k)) = V(w(k+1)) - V(w(k)) < 0 \quad \forall w(k) \in W, w(k) \neq 0, \]

we know that all initial states \( w(0) \in W \) belong to the basin of attraction \( B_A \) of the stabilized equilibrium point \( w = 0 \) (see, e.g., [15], p. 62).

Inserting (8) into the Lyapunov function (9) gives

\[ \Delta V(w(k)) = V(w(k+1)) - V(w(k)) = w(k+1)^T P w(k+1) - w(k)^T P w(k) = -w(k)^T Q(w(k)) w(k), \]

where \( Q \) is defined via

\[ (H_L + H_N(w(k)))^T P (H_L + H_N(w(k))) - P = -Q(w(k)). \tag{10} \]

It is easy to see that \( \Delta V(w(k)) < 0 \) if \( Q(w(k)) > 0 \).

For \( w(k) = 0 \), Eq. (10) simplifies to

\[ H_L^T P H_L - P = -Q(0), \tag{11} \]

which is the standard Lyapunov equation of discrete-time linear systems [13,16]. Since all the eigenvalues of \( H_L \) have modulus less than one, (11) has an unique solution \( P = P^T > 0 \) for every given positive-definite symmetric matrix \( Q(0) \) (see, e.g., [13], pp. 177-179).

Combining (10) and (11) and abbreviating \( w(k) \) by \( w \), we obtain the matrix equation

\[ Q(w) = Q(0) - H_N(w)^T P H_L - H_L^T P H_N(w) - H_N(w)^T P H_N(w). \tag{12} \]

Next, we will establish a criterion by means of which we can find a \( w \)-region about \( w = 0 \) where the matrix \( Q(w) \) remains positive-definite. This \( w \)-region will serve as the SSRC \( W \) introduced earlier.

According to Sylvester’s criterion ([13], p. 667), the matrix \( Q(w) \) is positive-definite if and only if its leading principal minors \( \Delta_i(w) \), that are the determinants of the \( i \times i \)-matrices in the top left-hand corner for \( i = 1, 2, \ldots, n \), are all positive. Since the matrix \( Q(0) \) is positive-definite, we have \( \Delta_i(0) > 0 \) for \( i = 1, 2, \ldots, n \). For \( w \neq 0 \), we need to check the validity of the \( n \) inequalities \( \Delta_i(w) > 0 \) for \( Q(w) \) to be positive-definite. Geometrically, speaking, these \( n \) inequalities constrain the size of the hyperellipsoid \( w^T P w = c \) for some \( c > 0 \), in the interior of which the condition \( Q(w) > 0 \) is satisfied. In order to maximize this hyperellipsoid, we need to maximize \( c \), which leads us to the constrained optimization problem

\[ \max_c c > 0 \text{ subject to } \Delta_i(w) > 0, \quad i = 1, 2, \ldots, n \quad \forall w : w^T P w \leq c. \tag{13} \]
We denote the solution of (13) by \( c_r \). The optimization problem (13) can be solved by sequential quadratic programming (SQP) [17].

Finally, the SSRC is obtained as

\[
W = \{ w \in \mathbb{R}^n | w^T P w \leq c_r \}.
\]  

(14)

The proposed method has an advantage over a direct numerical stability test, i.e., a brute force simulation of the controlled system using a discrete set of initial states situated on an \( n \)-dimensional grid of points. The number of initial states which would have to be tested in such a numerical stability test increases exponentially with the dimension \( n \) of the system under consideration. Moreover, the stability of the feedback system with an initial state different from the tested ones cannot be guaranteed. By contrast, the Lyapunov approach presented above ensures stability for all points within the SSRC \( W \). The evolutionary algorithm of Section 3 for computing an optimized SSRC has a search space of dimension \( \frac{1}{2} n(n + 1) \), which increases quadratically with \( n \). In addition, the design delivers the SSRC \( W \) by Eq. (14) in an analytic form easily implementable in the control algorithm.

We conclude this subsection by explaining how to cope with inputs constraints

\[
\| u(k) - \bar{u} \| \leq \delta
\]  

(15)

for some \( \delta > 0 \) and all \( k = 1, 2, \ldots \). In such a case, since

\[
u(k) - \bar{u} = -G w(k) \quad \text{for} \quad w(k) \in W,
\]  

(16)

we propose to shrink the size of \( W \) which is determined by the parameter \( c_r \) in (14). We have

\[
\| u(k) - \bar{u} \| \leq \| G \| \| w(k) \| = \sqrt{\text{eig}_{\text{max}}(G G^T)} \| w(k) \| \leq \left( \frac{\text{eig}_{\text{max}}(G G^T) c_r}{\text{eig}_{\text{min}}(P)} \right)^{1/2}.
\]

If

\[
\frac{\text{eig}_{\text{max}}(G G^T)}{\text{eig}_{\text{min}}(P)} c_r \leq \delta^2,
\]

then the input constraints (15) play no role. Otherwise (15) can be satisfied if \( c_r \) is replaced by

\[
c'_r = \delta^2 \frac{\text{eig}_{\text{min}}(P)}{\text{eig}_{\text{max}}(G G^T)}.
\]  

(17)

In the remainder of this paper we will work with \( c_r \), assuming that such a replacement is carried out if necessary under input constraints.

2.2. Controlling the Hénon map

The Hénon map [19] is a two-dimensional map defined by

\[
x(k + 1) = \left( \begin{array}{c} u(k) - x_1(k)^2 + 0.3 x_2(k) \\ x_1(k) \end{array} \right).
\]  

(18)

The bifurcation parameter \( u \in \mathbb{R} \) is regarded as the input. By setting \( x(k + 1) = x(k) \) in (18) we get the equilibrium points of the system as

\[
\tilde{x}^{(1)} = \left[ \frac{1}{2} (-0.7 + \sqrt{0.49 + 4 \bar{u}}), \frac{1}{2} (-0.7 + \sqrt{0.49 + 4 \bar{u}}) \right]^T,
\]

\[
\tilde{x}^{(2)} = \left[ \frac{1}{2} (-0.7 - \sqrt{0.49 + 4 \bar{u}}), \frac{1}{2} (-0.7 - \sqrt{0.49 + 4 \bar{u}}) \right]^T.
\]
The Taylor series expansion about $\tilde{x}$ leads to a system representation of the form of (7), where

$$A = \begin{pmatrix} -2\tilde{x}_1 & 0.3 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad G = (g_1, g_2), \quad h(w(k)) = \begin{pmatrix} -w_1(k) \\ 0 \end{pmatrix}.$$ 

In (8), we have

$$H_L = \begin{pmatrix} -2\tilde{x}_1 - g_1 & 0.3 - g_2 \\ 1 & 0 \end{pmatrix}, \quad H_N(w(k)) = \begin{pmatrix} -w_1(k) & 0 \\ 0 & 0 \end{pmatrix}.$$ 

The choice of the state feedback vector $G$ has to ensure that the eigenvalues of $H_L = A - BG$ have modulus less than one. So, $H_L$ is stable and there exists a symmetric positive-definite matrix $P$ which uniquely solves the equation $H_L^T P H_L - P = -Q(0)$, where $Q(0)$ is an arbitrary symmetric positive-definite matrix. We set $Q(0) = I$. Next, we apply Sylvester’s criterion to the matrix $Q(w) = Q(0) - H_N^T P H_L - H_L^T P H_N - H_N^T P H_N$ to obtain the constraints of the optimization problem (13), which is to be solved by SQP techniques, and finally we get $c_r$.

For the nominal input $\tilde{u} = 1.4$ the calculated quantities are summarized in Table 1. Fig. 2 shows the chaotic attractor of the Hénon map and the SSRCs about the stabilized equilibrium points $\tilde{x}^{(1)} = (0.8839, 0.8839)^T$ and $\tilde{x}^{(2)} = (-1.5839, -1.5839)^T$. We have used two different vectors $G$: the first one, as can be seen in Table 1, places the two eigenvalues of $H_L$ at the stable eigenvalue $\text{eig}_s(A)$ of $A$ and at zero (a choice which has been considered to be optimal in [2]), and the second value of $G$ makes both eigenvalues of $H_L$ equal to zero. Note that for both equilibrium points the SSRC $W$ and the chaotic attractor $A_R$ overlap, condition (6) holds, and hence the equilibrium point $\tilde{x}^{(1)}$ within the chaotic attractor as well as the equilibrium point $\tilde{x}^{(2)}$ outside the chaotic attractor can be stabilized. It can also be seen that the volumes of the SSRCs calculated with $\text{eig}(H_L) = (0, 0)$ are slightly larger than those with $\text{eig}(H_L) = (\text{eig}_s(A), 0)$, which raises the question as to how a particular choice of $G = (g_1, g_2)$
Fig. 2. The chaotic attractor $A_R$ of the Hénon map (18) and the SSRCs $W$ about the equilibrium points $\hat{x}^{(1)} = (0.8839, 0.8839)^T$ and $\hat{x}^{(2)} = (-1.5839, -1.5839)^T$ indicated by crosses. The SSRCs for $\text{eig}(H_L) = (0, 0)$ and $\text{eig}(H_L) = (\text{eig}_A, 0)$ are depicted by solid and dotted lines, respectively.

affects the volume of the SSRC. It is not difficult to see that $H_L$ is stable if $g_1, g_2$ take values within a triangular subset of the $(g_1, g_2)$ plane as introduced in Fig. 3. The volume $\text{Vol}$ of the SSRC $W = \left\{ w \in \mathbb{R}^2 | w^T P w < c_r \right\}$ is given by $\text{Vol} = \pi \left[ c_r / (\text{eig}_1(P) \text{eig}_2(P)) \right]^{1/2}$. We calculated the volume of the SSRC for the equilibrium point $\hat{x}^{(1)}$ on a gridded $(g_1, g_2)$ parameter plane for $Q(0) = I$. The result is shown in Fig. 3a; there is no SSRC associated with the points outside the triangle of stability, i.e., $\text{Vol} = 0$. It can be seen that the $\text{Vol}$ has a maximum at $g_1 = -1.7678$, $g_2 = 0.30$ which corresponds to $\text{eig}(H_L) = (0, 0)$. A similar analysis was conducted for the equilibrium point $\hat{x}^{(2)}$, and again the SSRC was largest when $\text{eig}(H_L) = (0, 0)$. The influence of different matrices $Q(0)$ on the volume of the SSRC can be seen from Fig. 3b and c: the volume depends strongly on the entries of $Q(0)$.

A numerical illustration of the control scheme is given in Fig. 4 where the $x_1$ coordinate is plotted as a function of discrete time. The control with $g_1 = -1.7678$ and $g_2 = 0.30$ was turned on at $k = 50$ to stabilize the equilibrium point $\hat{x}^{(1)}$. At $k = 100$ the control was switched to $g_1 = 3.1678$ and $g_2 = 0.30$, which eventually stabilized the system in the equilibrium point $\hat{x}^{(2)}$.

2.3. Controlling higher-dimensional systems

In recent years, the problem of extending the control of chaos to higher-dimensional systems has received a lot of attention (see, e.g., [20, 21]). We will now demonstrate that the proposed method can be used to stabilize equilibrium points of higher-dimensional systems with possibly more than one positive Lyapunov exponent. Let us consider the
Fig. 3. The two-dimensional volume of the SSRC $W$ of the equilibrium point $\bar{x}^{(1)} = (0.8839, 0.8839)^T$ depending on the parameters $g_1$ and $g_2$ for different $Q(0)$. The interior of the triangle with the boundary lines $g_2 = 1.3$, $g_2 = -g_1 - 0.7 - 2\bar{x}$ and $g_2 = g_1 - 0.7 + 2\bar{x}$ defines the region in the $(g_1, g_2)$ plane where the matrix $H_L = A - BG$ is stable.

generalized Hénon map [22]

$$x(k + 1) = \begin{pmatrix}
  u(k) - x_3(k)^2 - 0.1x_4(k) \\
  x_1(k) \\
  x_2(k) \\
  x_3(k)
\end{pmatrix}.$$  \hspace{1cm} (19)

For a brief study of this map, see Appendix A.
Fig. 4. Stabilization of the equilibrium points $N_{x.1}/D_{0}=0.8839; 0/8839/T$ and $N_{x.2}/D_{-1}=5839; -1/5839/T$. The control for the first equilibrium point was turned on at $k=50$ and switched to the other equilibrium point at $k=100$.

Assuming $u(k) = \bar{u} = \text{constant}$, the equilibrium points are

$$\bar{x}^{(1,2)} = \frac{1}{2}(-1.1 \pm \sqrt{1.21 + 4\bar{u}})(1, 1, 1)^T.$$  

The system (19) exhibits chaotic behavior for $\bar{u} = 1.76$; with Lyapunov exponents $\lambda_1 = 0.1538, \lambda_2 = 0.1418, \lambda_3 = 0.1189, \lambda_4 = -2.7171.$
We now turn to the design of the local control and expand the right-hand side of (19) into a Taylor series about $x$, introducing state feedback at the same time:

$$A = \begin{pmatrix} 0 & 0 & -2\tilde{x}_3 & -0.1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad G = (g_1, g_2, g_3, g_4),$$

$$h(w(k)) = \begin{pmatrix} -w_3(k)^2 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$ 

Next, we rewrite the system in the form of (8) where

$$H_L = A - BG = \begin{pmatrix} -g_1 & -g_2 & -2\tilde{x}_3 & -g_3 & -0.1 & -g_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$H_N(w(k)) = \begin{pmatrix} 0 & 0 & -w_3(k) & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$ 

The state feedback vector $G = (g_1, g_2, g_3, g_4)$ is calculated as described in Section 2.1, by first solving (11) with $Q(0) = I$. The results are summarized in Table 2. The equilibrium point $\tilde{x}^{(2)}$ cannot be stabilized since $W$ and $A_R$ have no common subset. In Fig. 5 a numerical test of the control scheme is given. The control is switched on at $k = 50$. It can be seen that it takes a considerable time until stabilization is achieved. This is typical for a vast majority of initial states $x_0$.

3. The evolutionary algorithm

3.1. The objective function

A characteristic feature of local control of chaos is that the length of a transient (between the starting point on the attractor or within its basin of attraction and the stabilized equilibrium point) depends sensitively on the position of the starting point [1,2]. Due to the ergodicity of chaotic dynamics this sensitivity can also be noticed by changing the switch-on time of the control scheme. The state of the system may be within the SSRC when the control is switching on. If not, a certain time elapses before the SSRC is entered. Apparently, the average time to achieve control is minimized if the overlap between the SSRC and the chaotic attractor is maximized.

Therefore, the following is suggested. The objective function to be maximized is the relative sojourn; i.e., the relative amount of time for which the trajectory on the chaotic attractor $A_R$ is within the SSRC $W$. This relative sojourn can be expressed as the measure (e.g., [23], p. 78):

$$\mu(A_R, W) := \lim_{k \to \infty} \frac{kW(A_R \cap W)}{K}.$$

(20)
Table 2
Numerical results obtained for local control of the four-dimensional generalized Hénon map (19) with nominal input $\bar{u} = 1.76$

<table>
<thead>
<tr>
<th>$\text{eig}(A)$</th>
<th>$\bar{\chi}^{(1)} = (0.8861, 1, 1, 1)^T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{eig}(H_0)$</td>
<td>$(0.6236 \pm 1.0485, -1.1907, -0.0564)$</td>
</tr>
<tr>
<td>$(g_1, g_2, g_3, g_4)$</td>
<td>$(-0.0564, 0, 0, 0)$</td>
</tr>
<tr>
<td>$(g_1, g_2, g_3, g_4)$</td>
<td>$(0.0564, 0, -1.7723, -0.10)$</td>
</tr>
</tbody>
</table>
| $P$ | \[
\begin{pmatrix}
4.0128 & 0 & 0 & 0 \\
0 & 3.0 & 0 & 0 \\
0 & 0 & 2.0 & 0 \\
0 & 0 & 0 & 1.0
\end{pmatrix}
\] |
| $Q(u)$ | \[
\begin{pmatrix}
1.0 & 0 & -0.2265w_3 & 0 \\
0 & 1.0 & 0 & 0 \\
-0.2265w_3 & 0 & -4.0128w_3^2 + 1.0 & 0 \\
0 & 0 & 0 & 1.0
\end{pmatrix}
\] |

$\Delta_1(u) = \Delta_2(u)$
$\Delta_3(u) = \Delta_4(u)$
$c_r$
$\text{Vol}$
\[
\begin{pmatrix}
4.0 & 0 & 0 & 0 \\
0 & 3.0 & 0 & 0 \\
0 & 0 & 2.0 & 0 \\
0 & 0 & 0 & 1.0
\end{pmatrix}
\]

$\Delta_1(u) = \Delta_2(u)$
$\Delta_3(u) = \Delta_4(u)$
$c_r$
$\text{Vol}$
\[
\begin{pmatrix}
1.0 & 0 & 0 & 0 \\
0 & 1.0 & 0 & 0 \\
0 & 0 & -4.0w_3^2 + 1.0 & 0 \\
0 & 0 & 0 & 1.0
\end{pmatrix}
\]

$\Delta_1(u) = \Delta_2(u)$
$\Delta_3(u) = \Delta_4(u)$
$c_r$
$\text{Vol}$
\[
\begin{pmatrix}
4.0 & 0 & 0 & 0 \\
0 & 3.0 & 0 & 0 \\
0 & 0 & 2.0 & 0 \\
0 & 0 & 0 & 1.0
\end{pmatrix}
\]

where $k_W(A_R \cap W)$ is the amount of time the trajectory on $A_R$ spends within $W$ during the discrete time interval $0 \leq k \leq K$.

Clearly, $\mu(A_R, W) = 1$ if $A_R \subseteq W$ and $\mu(A_R, W) = 0$ if $A_R \cap W = \emptyset$. Note that a positive measure $\mu(A_R, W) > 0$ implies condition (6). The measure $\mu(A_R, W)$ does not only represent the relative sojourn, it can also be interpreted as the geometric probability that the state of the system is in the SSRC at an arbitrary point in time. To get an approximate value of the objective function, the measure $\mu$ is replaced by $k_W(A_R \cap W)/K$ with $K$ sufficiently large.

It should be noted that the problem of minimizing the transient time of a trajectory to enter the SSRC can also be tackled by targeting (see, e.g., [24–27]). The difference to the presented approach is that targeting searches for an open-loop control, i.e., a pre-calculated control sequence

$$u = (\bar{u}^{(1)}, \bar{u}^{(2)}, \ldots, \bar{u}^{(l)})$$

which steers the system on the chaotic attractor to a target in minimal time using $l$ (possibly strongly bounded) inputs. By contrast, we attempt to find an optimized closed-loop controller that is effective in a maximal SSRC. So, we can combine the advantages of feedback control (these being, in general, a reduced sensitivity to modeling errors and noise [27]) with minimizing the time to achieve control.
Since the chaotic attractor $A_R$ is given by the uncontrolled system (1), a maximization of (20) can only be achieved by changing the SSRC $W$. In Section 2 a Lyapunov approach to local control was presented by means of which the SSRC can be determined depending on the adjustable controller parameters $Q(0)$ and $G$. The entries of both of these matrices are arbitrary up to the conditions that $Q(0)$ is positive-definite and the eigenvalues of $A - BG$ have modulus less than one. Previously, an optimal choice of the feedback gain matrix $G$ was investigated [2,5]. Here, we fix the feedback gain matrix $G$ and optimize over the controller parameter $Q(0)$.

3.2. Basic structure of the evolutionary algorithm

In the following we will generalize the measure $\mu(A_R, W)$ by varying the elements of the matrix $Q(0)$ and fixing $G$ at their optimal entries. As in Section 2, we obtain the SSRC $W = W(Q(0))$ from the Lyapunov equation (11), the subsequent solution of the optimization problem (13), and from (14). The constrained optimization problem under consideration reads

$$\max_{Q(0)} \mu(A_R, W(Q(0)) \quad \text{subject to} \quad Q(0) = Q(0)^T > 0.$$  \hspace{1cm} (21)

The optimized $Q(0)$ shall be denoted by $Q_s$, i.e.,

$$Q_s = \arg \max_{Q(0)} \mu(A_R, W(Q(0)) \quad \text{subject to} \quad Q(0) = Q(0)^T > 0.$$  

The subsequently calculated (optimized) quantities shall be called $P_s$ and $c_s$. Thus, the optimized SSRC is $W = \{w \in \mathbb{R}^n \mid w^T P_s w < c_s\}$. As mentioned before, solving (21) is difficult since the objective function $\mu(A_R, W)$ is not an analytic function of the adjustable entries in $Q(0)$. Hence, both the SSRC $W$ depending on $Q(0)$ and the objective function $\mu(A_R, W)$ can only be determined numerically. Moreover, $\mu(A_R, W)$ may have discontinuities.
and regions where the objective function is nearly constant, see Fig. 2. The figure also shows that the objective function might be multi-modal, i.e., it may have several local optima. Furthermore, it is conceivable that as the size or shape of the SSRC changes, it might be that the SSRC will include a not directly connected subset of the attractor which was not lying in the SSRC before.

These considerations suggest that some optimization methods, in particular those requiring certain smoothness conditions, are less suitable. A better way to solve the optimization problem (21) appears to be the use of an evolutionary algorithm similar to the ones in [9,10,28].

The evolutionary algorithm we will employ has the following structure.

**Algorithm.**

(i) Create a set $M^1$ of symmetric, positive-definite matrices $Q(0)$ which are used to determine the objective function $\mu(A_R, W(Q(0)))$ for all $Q(0) \in M^1$. Their entries are initialized randomly and should cover the search space. Put $k := 1$.

(ii) Calculate the objective function $\mu(A_R, W(Q))$ for every matrix $Q$ of the set $M^k$.

(iii) Select matrices for which the value of the objective function is above average.

(iv) Create a new set from matrices chosen in (iii).

(v) Perform random alterations in the elements of the matrices created in (iv). The resulting set of matrices is called $M^{k+1}$.

(vi) Increase $k$ by 1 and loop to (ii) unless a maximal value of the objective function or another termination criterion is reached.

Since such an optimization algorithm bears some features of natural evolution, the set of matrices $M^k$ is frequently called the population of individuals at generation $k$, the objective function $\mu$ is termed fitness, and instead of “creation” and “alteration” as in (iv) and (v) one speaks of recombination and mutation (see, e.g., [9,10,18]). In the sequel, we will use these conventions, too.

The above algorithm can also be regarded as a probabilistic dynamical system mapping $M^k$ onto $M^{k+1}$ by means of the operators selection, recombination and mutation depending on the fitness values of the elements of $M^k$ as well as on random variables. Here, the mean and maximal fitness value associated with the set $M^{k+1}$ are meant to increase with respect to the fitness values of $M^k$ (by selecting matrices whose fitness is above average), but since the algorithm is probabilistically driven, such an increase need not occur at every step $k$.

It should be mentioned that this kind of an algorithmic structure is also realized in genetic algorithms [18,29]. The main difference to the evolutionary algorithm proposed here is the different representation of the objects to optimize. While in evolutionary algorithms the objects to optimize remain in their natural form as points in a search space represented by vectors or matrices with real entries, genetic algorithms use a binary coding (see, e.g., [9,10,28] for a detailed discussion of differences and similarities).

### 3.3. Representation, initialization, and operators

This subsection explains the steps (i)-(vi) of the algorithm given above (see also Fig. 6).

#### 3.3.1. Representation and initialization

The set of “candidate” matrices to be optimized over is supposed to consist of $l$ matrices at every discrete time step $k = 1, 2, \ldots, K$:

$$M^k = (Q_1^k \quad Q_2^k \quad \cdots \quad Q_l^k),$$
where

\[ Q^k_i \in \mathbb{R}^d, \quad i = 1, 2, \ldots, l, \quad d = \frac{1}{2}n(n + 1). \]

Here, the \( n \times n \)-matrices \( Q^k_i \) may be seen as matrices \( Q(0) \) at step \( k \). They are symmetric and positive-definite. For initialization symmetric matrices are generated whose entries are realizations of a random variable normally distributed on \([0, \gamma]\). Matrices which are positive-definite are included into \( M^1 \). To make the calculation of the operators selection, recombination and mutation easier the number \( l \) is chosen as an even number. As a result of this representation, we have a search space of \( \frac{1}{2}n(n + 1) \) in which \( l \) individuals are supposed to evolve toward the optimum.

3.3.2. Fitness

To measure the fitness of each matrix \( Q^k_i (i = 1, 2, \ldots, l; k = 1, 2, \ldots, K) \), we associate a real number

\[ \mu^k_i \in [0, 1] \]

with each \( Q^k_i \), which is calculated as before: given \( Q^k_i \), we use (11) to compute \( P^k_i \), (13) to determine \( c^k_i \), (14) to calculate \( W^k_i \), and \( \mu^k_i \) is finally given by (20).

3.3.3. Selection

The selection operator

\[ s : \mathbb{R}^b \times \mathbb{Z}^l \to \mathbb{R}^h, \quad h = ld \]

produces the set \( M^k_s \) from the set \( M^k \) and its \( l \) fitness values \( \mu^k_i \). For this purpose, a matrix pair \( (Q^k_i, Q^k_j) \) is taken from \( M^k \) at random \( l \) times where the subscripts \( i, j, i \neq j \) are independently chosen as realizations of an integer random variable uniformly distributed on \([1, l]\). If \( \mu^k_i \geq \mu^k_j \) then \( Q^k_i \) is included in \( M^k_s \), otherwise \( Q^k_j \).

This selection procedure (sometimes called tournament selection) imposes a linear dependence between the probability \( p_s(Q^k_i) \) of a matrix \( Q^k_i \) being chosen and its fitness such that the matrix with lowest fitness has probability 0 and the matrix with highest fitness is most likely to be inserted into \( M^k_s \) [18].
3.3.4. Recombination

The recombination operator

\[ r : \mathbb{R}^b \times \mathbb{R}^d \to \mathbb{R}^d \]

produces a set \( M_k^b \) from the elements of \( M_k^d \) as in Section 3.3.3. \( l \) random choices of the matrix pairs \((Q^i_k, Q^j_k)\) are made, now with \( Q^i_k, Q^j_k \) being elements of \( M_k^d \). Each time a new matrix \( \hat{Q} := \{ \mu_i^k / (\mu_i^k + \mu_j^k) \} Q^i_k + \{ \mu_j^k / (\mu_i^k + \mu_j^k) \} Q^j_k \) is calculated, and inserted into \( M_k^b \). This rule for calculating \( \hat{Q} \) is motivated by the requirement that a matrix from \( M_k^b \) (and therefore from \( M_k^d \)) with a higher fitness should have more influence on the elements of \( M_k^b \) (and hence on \( M^{k+1} \)) than a matrix with lower fitness. Since \( \mu_i^k / (\mu_i^k + \mu_j^k) > 0 \) and \( \mu_j^k / (\mu_i^k + \mu_j^k) > 0 \), it is guaranteed that all matrices from \( M_k^b \) are again symmetric and positive-definite.

3.3.5. Mutation

The mutation operator

\[ m : \mathbb{R}^d \to \mathbb{R}^d \]

is applied to every matrix \( Q^k \) in \( M_k^d \). It randomly performs one of the following operations with the probabilities \( p_a, p_b \) and \( p_c \), where \( p_a + p_b + p_c = 1 \).

(a) The matrix \( Q^k \) remains unchanged.

(b) A diagonal element of \( Q^k \) is changed. An entry \( q_{jj} \) of \( Q^k \) is updated as follows: the index \( j \) is a realization of an integer random variable uniformly distributed on \([1, n]\). Then, a realization of a random variable normally distributed on \([0, \alpha]\) is added to the previous value of \( q_{jj} \).

(c) An off diagonal element of \( Q^k \) is changed. For an entry \( q_{jh} \) of \( Q^k \) to be updated, the indices \( j, h \) are first independently chosen as realizations of an integer random variables uniformly distributed on \([1, n]\). Then, a realization of a random variable normally distributed on \([0, \alpha]\) is added to the previous value of \( q_{jh} \). To preserve the symmetry of the updated matrix \( \hat{Q}^k \), we set \( q_{hj} = q_{jh} \).

The alterations (b) and (c) require to check that the resulting matrix is still positive-definite. If this is not achieved, the random action is repeated until the updated matrix becomes positive-definite. After the mutation operator is applied to all matrices of \( M_k^d \), the stored best matrix is inserted by replacing a randomly chosen matrix. This set is called \( M^{k+1} \); the procedure starts again with the evaluation of fitness as given in Section 3.3.2.

4. Numerical results

4.1. Optimizing the control of the Hénon map

The optimization algorithm proposed in Section 3.2 is now applied to the local control of the Hénon map (18), and the results are summarized in Figs. 7–11. The algorithm was tested for two different feedback vectors \( G = (g_1, g_2) \), each of which remained constant during the execution of the algorithm, and for both of the equilibrium points of the Hénon map. The optimized SSRCs given by \( Q_c, P_c \) and \( c_s \) and characterized by the fitness measure \( \mu \) are listed in Table 3 and compared with the results \( P, c_r \) and \( \mu \) obtained for \( Q(0) = I \).

The parameters of the algorithm are the maximum number of generations \( K \), the size of population \( l \) and the probabilities \( p_a, p_b \) and \( p_c \) as well as the weight \( \alpha \) of the mutation operator. The following parameter values were chosen: \( K = 100, l = 24, \alpha = 1025, p_a = 0.50 \) and \( p_b = 0.35, p_c = 0.15 \) for the equilibrium point \( \tilde{x}^{(1)} \), whilst \( p_b = 0.15, p_c = 0.35 \) for the equilibrium point \( \tilde{x}^{(2)} \). The algorithm was initialized by creating \( l \) symmetric positive-definite \( 2 \times 2 \) matrices out of \([0, 1]\) normally distributed random numbers which were multiplied by 10,000.
Fig. 7. The maximal (solid line) and mean value (dotted line) of the fitness of the Hénon map optimization as a function of the generation $k$.
(a) The equilibrium point $\tilde{x}_1 = (0.8839, 0.8839)$ and $g_1 = -1.7678$, $g_2 = 0.30$. (b) The equilibrium point $\tilde{x}_2 = (-1.5839, -1.5839)$ and $g_1 = 3.1678$, $g_2 = 0.30$. 
Fig. 8. The maximal (solid line) and mean value (dotted line) of the volume of the SSRC as a function of the generation $k$, where $\bar{x}^{(2)} = (-1.5839, -1.5839)$ and $g_1 = 3.1678, g_2 = 0.30$.

Fig. 9. The chaotic attractor of the Hénon map and the obtained SSRCs for $\text{eig}(H_L) = (0, 0)$. 
Fig. 10. The maximal (solid line) and mean value (dotted line) of the fitness of the Hénon map optimization as a function of the generation $k$.
(a) The equilibrium point $\bar{x} = (0.8839, 0.8839)$ and $g_1 = -1.9237, g_2 = 0.30$. (b) The equilibrium point $\bar{x} = (-1.5839, -1.5839)$ and $g_1 = 3.2598, g_2 = 0.30$. 
Fig. 11. The chaotic attractor of the Hénon map and the obtained SSRC for \( \text{eig}(H_L) = (\text{eig}(A), 0) \).

Fig. 7a shows the results of the optimization for \( g_1 = -1.7678 \) and \( g_2 = 0.30 \) and the equilibrium point \( \tilde{x}^{(1)} = (0.8839, 0.8839)^T \). Fig. 7b displays analogous results for the equilibrium point \( \tilde{x}^{(2)} = (-1.5839, -1.5839)^T \), with \( g_1 = 3.1678 \) and \( g_2 = 0.30 \). In Fig. 8 it is depicted how the volume of the SSRC evolves in time. This algorithm run is an illustrative example for the evolutionary algorithm’s ability to overcome local optima. It can also be seen that the relation between \( Q(0) \) and the measure \( \mu \) is discontinuous, if we compare Figs. 7 and 8. Fig. 9 displays the chaotic attractor of the Hénon map and shows the SSRCs about the two equilibrium points for \( Q(0) = I \) and for the solution of the evolutionary algorithm \( Q(0) = \bar{Q}_s \). For \( Q(0) = I \) the SSRCs contains just parts of one branch of the chaotic attractor. During the execution of the algorithm the volume of the SSRC \( W \) increases. This increase in volume brings about only a minor increase of the measure \( \mu \) at the beginning. But as the SSRC becomes large enough to include the other branch of the chaotic attractor, too, the measure \( \mu \) increases discontinuously.

Similar results were obtained for the other parameter set (see Fig. 10). The results for \( g_1 = -1.9237 \), \( g_2 = 0.30 \) and both equilibrium points are also given in Table 3, and a comparison of the SSRCs obtained for \( Q(0) = I \) and \( Q(0) = Q_s \) is depicted in Fig. 11. To conclude, we noted that a significant increase of the measure \( \mu \) (and hence a significant decrease of the average time to achieve control) can be achieved by finding an optimized \( Q(0) \).

4.2. Optimizing the control of the four-dimensional map

In Section 2.3 the four-dimensional Hénon map (19) was studied. For some nominal input \( \bar{u} \) the system exhibits chaotic behavior and has three positive Lyapunov exponents. A local controller was designed which stabilizes an unstable equilibrium point embedded in the chaotic attractor. However, numerical experiments have shown that the
transient time from the chaotic attractor to the stabilized equilibrium point can be considerable. This is caused by a comparatively small SSRC $W$ and, more importantly, by a small measure of the uncontrolled chaotic attractor within the SSRC. In the following we will attempt to maximize the measure using the evolutionary algorithm presented in Section 3.

The parameters in the algorithm were fixed at $\gamma = 20\,000$, $K = 500$, $l = 48$, $p_a = 0.3$, $p_b = p_c = 0.35$, $\alpha = 12\,500$. The results of the optimization and a comparison with the case $Q(0) = I$ are listed in Table 4. Fig. 12a shows the evolution for $g_1 = g_2 = 0$, $g_3 = -1.7723$, $g_4 = -0.10$ with the maximal and mean value of the measure $\mu$ being depicted. Although the population is chosen to be much larger than for the two-dimensional example, mean and maximal value of fitness start from a much lower level. This becomes plausible taking into account that the likelihood to find a matrix $Q(0)$ with high fitness at random scales with the dimension of search space, $\frac{1}{2}n(n+1)$. This also indicates that a pure random search for $Q(0)$ (i.e., to generate matrices $Q(0)$ from random variables and comparing them) is not likely to be successful for higher-dimensional systems. Fig. 13 shows how the volume of the SSRC evolves in time. It can be seen that from time to time a large outburst in the maximal value of the volume occurs, indicating that an SSRC with a much larger volume had been produced. Most of the time this jump in SSRC volume does not cause the measure $\mu$ to increase as well, and hence the associated matrix $Q(0)$ is not likely to be kept in the next step. Finally, similar results for $g_1 = -0.0546$, $g_2 = 0$, $g_3 = -1.7723$, $g_4 = -0.10$ are shown in Fig. 12b and Table 4. By comparing the optimization results for both examples, it is observed that for higher-dimensional systems the computations usually take longer and a larger population $l$ is needed to get the desired behavior of convergence.

<table>
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<tr>
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<th>$\lambda^{(2)} = (-1.5839, -1.5839)^T$</th>
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</tr>
<tr>
<td>$0$ $1.0$</td>
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Fig. 12. The maximal (solid line) and mean value (dotted line) of the fitness for optimizing the local control of the four-dimensional Hénon map as a function of the generation $k$. (a) Parameters $g_1 = g_3 = 0$, $g_3 = -1.7723$, $g_4 = -0.10$. (b) Parameters $g_1 = -0.0564$, $g_2 = 0$, $g_3 = -1.7723$, $g_4 = -0.10$. 
Table 4

Results of the evolutionary algorithm for optimizing the local control of the four-dimensional generalized Hénon map (19)

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5. Concluding remarks

In this paper, optimizing the local control of chaos by means of an evolutionary algorithm was investigated. It was the average time to achieve control which we wanted to minimize, and thus we tried to maximize the relative sojourn of the trajectory on the chaotic attractor within the SSRC. Based on a Lyapunov approach to local control of chaos, the SSRC \(W\) was determined using a pre-specified positive-definite matrix \(Q(0)\) and a feedback gain matrix \(G\). The feedback gain matrix \(G\) can be obtained by an optimization using standard linear control theory (see, e.g., [5]). In addition, we observed that \(G\) is of little influence on size and shape of the SSRC. Therefore, we assumed the feedback law to be given and optimized over the controller parameter \(Q(0)\). Since the relation between adjustable parameters in the matrix \(Q(0)\) and the relative trajectory sojourn appeared to be too complicated to deal with by
traditional optimization methods, an evolutionary optimization algorithm was used. The method proposed in this paper was tested on the standard two-dimensional Hénon map as well as on a four-dimensional generalized Hénon map taken from [22].

Appendix A. The generalized four-dimensional Hénon map

A four-dimensional generalization of the standard Hénon map (18) is given by

$$x(k + 1) = \begin{pmatrix} u(k) - x_3(k)^2 - bx_4(k) \\ x_1(k) \\ x_2(k) \\ x_3(k) \end{pmatrix}, \quad (A.1)$$

where $x \in \mathbb{R}^4$, $u \in \mathbb{R}$, and $b > 0$, see [22].

For a nominal input $u(k) = \bar{u} > 0$, the equilibrium points $\bar{x}$ of the map (A.1) are

$$\bar{x}^{(1,2)} = \frac{1}{2}[-(1 + b) \pm \sqrt{(1 + b)^2 + 4\bar{u}}](1, 1, 1)^T.$$

The stability of the equilibrium points can be determined by evaluating the eigenvalues of the Jacobian

$$A = \begin{pmatrix} 0 & 0 & -2\bar{x}_3 & -b \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \quad (A.2)$$

Note that $\det A = b$ and hence the system is dissipative for $b < 1$. 
The characteristic polynomial of the linearized open-loop system is

\[ P_O(z) = \det(zI - A) = z^4 + 2\bar{x}_3 z + b. \]  

The equilibrium point \( \bar{x} \) is locally stable if all the roots of the characteristic polynomial have modulus less than one, which can be assessed using the Cohn–Schur criterion, see, e.g. [30]. Taking into account that \( b > 0 \) was assumed, we find that stability occurs if and only if the inequality

\[ 4\bar{x}_3^2 < (1 - b)^2(1 + b) \]

is fulfilled.

The equilibrium point \( \bar{x}^{(1)} = \frac{1}{2}[-(1 + b) + ((1 + b)^2 + 4\bar{u})^{1/2}](1, 1, 1)^T \) is locally stable for

\[ 0 < \bar{u} < \frac{1}{4}(1 - b^2)(1 - b + \sqrt{1 + b}). \]  

(A.4)

By contrast, the equilibrium point \( \bar{x}^{(2)} = \frac{1}{2}[-(1 + b) - ((1 + b)^2 + 4\bar{u})^{1/2}](1, 1, 1)^T \) is unstable for all \( \bar{u} \).

From now on (as in Section 2.3) we keep the parameter \( b \) fixed at the value \( b = 0.1 \).

Next, we look into how the attractor of the map (A.1) changes with the nominal input \( \bar{u} \). As derived above, the equilibrium point \( \bar{x}^{(1)} \) is locally stable for \( \bar{u} < 0.7419 \). For \( \bar{u} = 0.7419 \) a Hopf-bifurcation occurs giving rise to a stable quasi-periodic orbit characterized by the largest Lyapunov exponent \( \lambda_1 = 0 \). Increasing \( \bar{u} \) further leads again to periodic motions and finally to chaotic behavior in which \( \bar{u} \)-intervals with one, two or three positive Lyapunov exponents are distinguishable. For \( \bar{u} > 1.77 \) the system is unstable. A bifurcation diagram illustrating this route to chaos is shown in Fig. 14.
To obtain a quantitative characterization of the attractors shown in the bifurcation diagram, the set of Lyapunov exponents $\Lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ was computed (see, e.g., [23,31]). The results are given in Fig. 15. A comparison with the bifurcation diagram shows clearly the $\bar{u}$-regions where the system exhibits chaotic behavior and at least one Lyapunov exponent is larger than zero.

Since the system is assumed to be exactly known, the Lyapunov exponents $\lambda_i$, $i = 1, \ldots, 4$ can be computed using standard methods. Note that $\sum_{i=1}^{4} \lambda_i = \ln |\det A| = \ln(0.1)$. It is observed that the Lyapunov exponents are the same for almost all initial conditions within the basin of attraction of the chaotic attractor. The Lyapunov exponents for the nominal input $\bar{u} = 1.76$ are listed in Table 5 (10,000 iterations, 256 different initial states).

<table>
<thead>
<tr>
<th>$\bar{u}$</th>
<th>$\lambda_1$</th>
<th>$\lambda_2$</th>
<th>$\lambda_3$</th>
<th>$\lambda_4$</th>
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<tr>
<td></td>
<td>0.1538</td>
<td>0.1418</td>
<td>0.1189</td>
<td>-2.7171</td>
</tr>
<tr>
<td>$\max_{1 \leq j \leq N_0} \lambda^{(j)}$</td>
<td>0.1570</td>
<td>0.1461</td>
<td>0.1254</td>
<td>-2.7028</td>
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<tr>
<td>$\min_{1 \leq j \leq N_0} \lambda^{(j)}$</td>
<td>0.1498</td>
<td>0.1362</td>
<td>0.1116</td>
<td>-2.7287</td>
</tr>
<tr>
<td>$\frac{1}{N_0} \sum_{j=1}^{N_0} (\lambda^{(j)} - \bar{\lambda})^2$</td>
<td>0.0015</td>
<td>0.0017</td>
<td>0.0027</td>
<td>0.0043</td>
</tr>
</tbody>
</table>

Table 5
Numerical calculation of the Lyapunov exponents of the four-dimensional generalized Hénon map with nominal input $\bar{u} = 1.76$ (10,000 iterations, $N_0 = 256$ different initial states).
References